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Balanced 0, ± Matrices
Part II: Recognition Algorithm

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Abstract

In this paper we give a polynomial time regocnition algorithm for balanced 0, ± matrices. This algorithm is based on a decomposition theorem proved in a companion paper.

1 Introduction

In [2], Conforti, Cornuéjols and Rao prove a decomposition theorem for balanced $0,1$ matrices and they use it to obtain a polynomial time recognition algorithm of these matrices. In this paper, using a similar approach, we give a polynomial time recognition algorithm for balanced $0,\pm1$ matrices, using a decomposition result derived in the companion paper [1]. In this paper, as in [1], we work on the signed bipartite graph representation of a $0,\pm1$ matrix. All relevant notation can be found in [1].

The decomposition theorem [1] uses two types of edge cutsets, namely 2-joins and 6-joins, and a certain kind of node cutset. When we remove the edges (nodes) of a cutset in a signed bipartite graph G, it is not true in general that, if the resulting connected components are balanced, then G is balanced. However we may be able to achieve this property by adding a few nodes and edges to the connected components. In Section 2 we give such a construction for the 2-joins and 6-joins. The situation for the node cutset is more complicated and is dealt with in Section 3. In Section 4 we give a polynomial time algorithm for identifying a 6-join and in Section 5 for identifying a 2-join. In Section 6 we put all the pieces together and give a polynomial algorithm for recognizing if a signed bipartite graph is balanced.

2 Edge Decompositions

Throughout this paper, we assume that G be a signed bipartite graph. The sides of the bipartition are V^c and V^r with $m = |V^r|$ and $n = |V^c|$. The *length* of a path P is the number of its edges and its *weight* $w(P)$ is the sum of its edge weights. Similarily we distinguish between the length and weight of a cycle. If the weight of a cycle is 0 mod 4, we say that the cycle is *quad*, otherwise it is *unquad*. By *scaling* G at node u, we mean changing the sign of the weights on all the edges incident with u.

Remark 2.1 *Let G' be a signed bipartite graph obtained from G by scaling at node u. A cycle C is quad in G' if and only if it is quad in G.*

G is *restricted balanced* if all its cycles are quad. We have the following version of Theorem 5.1 in [1].

Theorem 2.2 *Let G be a signed bipartite graph. If G is balanced but not restricted balanced then either the underlying graph is R_{10} or G contains a 2-join, a 6-join or an extended star cutset.*

2-Join Decomposition

Let $E(K_{BD}) \cup E(K_{EF})$ be a 2-join and G_{BE} (G_{DF}) the union of the components of $G \setminus E(K_{BD}) \cup E(K_{EF})$ containing a node of B (a node of D). Recall that, according to our definition of a 2-join in Part I [1], G_{BE} contains E and G_{DF} contains F. When neither $D \cup F$ nor $B \cup E$ induces a biclique, we construct the *block* G_1 from G_{BE} as follows:

- Add two nodes d and f, connected respectively to all nodes in B and to all nodes in E.
- Let P_2 be a chordless path in G_{DF} connecting a node $d' \in D$ to a node $f' \in F$. If $w(P_2) \equiv 0 \bmod 4$ or $w(P_2) \equiv 2 \bmod 4$, nodes d and f are connected by a path of length 4 of weight 0 or 2 respectively. If $w(P_2) \equiv 1 \bmod 4$ or $w(P_2) \equiv 3 \bmod 4$, nodes d and f are connected by a path of length 5 of weight 1 or 3 respectively. Denote this path by P_{df}. Sign the edges between node d and the nodes in B exactly the same as the corresponding edges between d' and the nodes of B in the original graph. Similarly, sign the edges between f and the nodes in E exactly the same as the corresponding edges between f' and the nodes in E.

The block G_2 is defined similarly from G_{DF}.

Remark 2.3 *If $E(K_{BD}) \cup E(K_{EF})$ is a 2-join and $B \cup E$ ($D \cup F$) induces a biclique, then $B \cup E$ ($D \cup F$) is a biclique cutset of G.*

Theorem 2.4 *Let G_1 and G_2 be the blocks of the decomposition of the signed bipartite graph G by a 2-join $E(K_{BD}) \cup E(K_{EF})$, such that neither $B \cup E$ nor $D \cup F$ induces a biclique. If $K_{BD} \cup K_{EF}$ is balanced, then G is balanced if and only if both G_1 and G_2 are balanced.*

The following lemma is used in the proof of Theorem 2.4.

Lemma 2.5 *Let G be a signed bipartite graph with no unquad hole of length four. For every biclique K_{BD} in G, we can scale G on the nodes in $B \cup D$ so that every edge in $E(K_{BD})$ has weight +1.*

Proof: If $|B| = 1$ then we can scale on nodes in D to obtain the result. Similarily, for $|D| = 1$.

We can assume $|B| \geq 2$ and $|D| \geq 2$. Let $b \in B$ and $d \in D$. Scale at nodes $d' \in D$ so that all edges (b, d') have weight $+1$. Scale at nodes $b' \in B$ so that all edges (b', d) have weight $+1$. Every $d' \in D \setminus \{d\}$ and $b' \in B \setminus \{b\}$ induce a hole b, d, b', d', b of length four. By assumption this hole is quad. Hence (b', d') must have weight $+1$. □

Remark 2.6 *Let G be a signed bipartite graph with no unquad hole of length 4. By Lemma 2.5 there exists a signed graph G', which is obtained from G by a sequence of scalings, such that all the edges in $E(K_{BD}) \cup E(K_{EF})$ have weight $+1$, since K_{BD} and K_{EF} are node disjoint.*

Proof of Theorem 2.4: By Remark 2.6 we can assume that all the edges in $E(K_{BD})$ and $E(K_{EF})$ have weight $+1$. First we show that G_1 and G_2 are balanced if G is balanced. Every hole H in G_1 corresponds to a hole H' in G, except for the case where H contains nodes d and f and no other nodes of P_{df}, and $D \cup F$ is a biclique in G. The existence of such a biclique would contradict our assumption. The hole H' has the same weight as H, since all the edges of $E(K_{BD}) \cup E(K_{EF})$ are all signed positive. Thus G_1 is balanced if G is balanced. Similarly for G_2.

Now assume that G_1 and G_2 are balanced, but G is not. Let H be an unquad hole of G. If it contains no edge of G_{DF}, there exists a hole in G_1 which is unquad. The same argument holds for G_{BE}.

Let $H = b', d', Q_2, f', e', Q_1, b'$ where $b' \in B, d' \in D, f' \in F$ and $e' \in E$ be an unquad hole in G. Since G_1 is balanced, $w(Q_2)$ and $w(P_{df})$ are not congruent modulo 4. But by defintion of a block, there exists a path P_2 from $d'' \in D$ to $f'' \in F$, such that $w(P_2)$ is congruent to $w(P_{df})$ modulo 4. The holes $H_1 = d'', P_2, f'', P_{eb}, b, d''$ and $H_2 = d', Q_2, f', e, P_{eb}, b, d'$ have distinct weights modulo 4. Hence one of them must be unquad, contradicting our assumption. □

6-Join Decomposition

Let $A_1, \ldots, A_6$ be disjoint nonempty node sets in the signed bipartite graph G such that the edges of the graph A induced by $\cup_{i=1}^{6} A_i$ form a 6-join. Let G_{135} be the union of the components of $G \setminus E(A)$ containing a node in $A_1 \cup A_3 \cup A_5$ and G_{246} the union of the components containing a node in $A_2 \cup A_4 \cup A_6$. We construct the *block* G_1 from G_{135} as follows:

- Add node a_2 and edges between a_2 and all the nodes in A_1 and A_3, node a_4 and edges between a_4 and all the nodes in A_3 and A_5 and node a_6 and edges between a_6 and all the nodes in A_5 and A_1.
- Pick any three nodes $a'_2 \in A_2$, $a'_4 \in A_4$ and $a'_6 \in A_6$ and sign the edges of G_1 connected to a_2, a_4 and a_6 according to the signs of the corresponding edges connected to a'_2, a'_4 and a'_6.

Similarly, the block G_2 is defined from G_{246}.

Theorem 2.7 *Let G_1 and G_2 be the blocks of the decomposition of the signed bipartite graph G by a 6-join $A = G(\cup_{i=1}^{6} A_i)$ such that A is balanced. Then G is balanced if and only if both G_1 and G_2 are balanced.*

We first prove the following lemma.

Lemma 2.8 *If A does not contain an unquad hole, then there exists a signing of G which is obtained by a sequence of scalings on the nodes of A, such that for every biclique $K_{A_iA_{i+1}}, i \in \{1, \ldots, 6\}$ (where indices are taken modulo 6) the edges in the biclique are all signed +1 or they are all signed -1.*

Proof: By Lemma 2.5 we can sign all the edges in $E(K_{A_1A_2}), E(K_{A_3A_4})$ and $E(K_{A_5A_6})$ to be +1. W.l.o.g. let $E(K_{A_2A_3})$ contain an edge signed +1 and another signed -1. Now there exist in A two holes of length 6 which differ in weight by 2. Clearly one of these must be unquad contradicting our assumption that A contains no unquad hole. □

Proof of Theorem 2.7: By Lemma 2.8 we can assume that for every biclique $K_{A_iA_{i+1}}, i \in \{1, \ldots, 6\}$, the edges of the biclique are all signed +1 or they are all signed -1.

It follows from the definition of the blocks that G_1 and G_2 are induced subgraphs of G and so are balanced if G is balanced.

Let H be an unquad hole of G. If it contains no edge of G_{246}, there exists a hole in G_1 which is unquad. The same argument holds for G_{135}.

Now we can assume that the hole has an edge in G_{135} and an edge in G_{246}. Clearly H must have exactly four nodes in common with $V(A)$ otherwise H contains a chord.

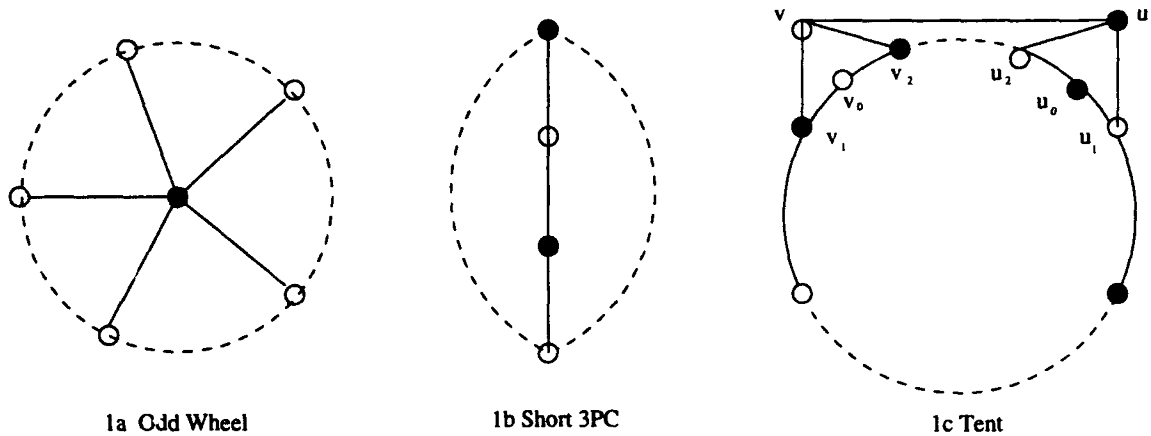

Figure 1: Odd wheel, short 3-path configuration and tent

W.l.o.g. let $H = a'_1, P_1, a'_5, a'_4, P_2, a'_2, a'_1$ where $a'_1 \in A_1, a'_2 \in A_2, a'_4 \in A_4$ and $a'_5 \in A_5$. Then either a_1, P_1, a_5, a_6, a_1 or a_4, P_2, a_2, a_3, a_4 is an unquad hole, otherwise by adding the weights of these disjoint holes and H, and observing that H is unquad we obtain that $a_1, a_2, a_3, a_4, a_5, a_6, a_1$ is an unquad hole contradicting our assumption. □

3 Double Star Decomposition

A *double star* is a node set $N(u) \cup N(v)$ where uv is an edge of the graph. Let S be an extended star cutset or a double star cutset of G and $G'_1, \ldots, G'_k$ the connected components of $G \setminus S$. We define the *blocks* of the decomposition to be signed bipartite graphs $G_1, \ldots, G_k$ where each of the blocks G_i is obtained by taking the induced signed subgraph on the node set $V(G'_i) \cup S$.

The extended star and double star decompositions are not balancedness preserving, i.e. the blocks $G_1, \ldots, G_k$ may be balanced even though the signed bipartite graph G is not. For example the graphs of Figure 1 are not balanced, but contain a double star cutset with resulting blocks that are balanced. Our recognition algorithm for the class of balanced signed bipartite graphs exploits the structure of signed bipartite graphs that are not balanced. Conforti and Rao [3] and later Conforti, Cornuéjols and Rao [2] have studied bipartite graphs that are not balanced. In the next section

this study is extended to *signed* bipartite graphs.

If the signed bipartite graph G is decomposed recursively using extended star decompositions on the blocks, we could end up using an exponential number of steps in the decomposition. Our recognition algorithm uses double star decompositions instead, for which we can prove that the number of steps is polynomial.

Definition 3.1 *A node u is said to be* dominated *if there exists a node v, distinct from u, such that $N(u) \subseteq N(v)$. A graph is said to be* undominated *if it contains no dominated nodes.*

Lemma 3.2 *[2] If an undominated bipartite graph contains an extended star cutset, then it contains a double star cutset.*

3.1 Smallest Unquad Holes

Assume the signed bipartite graph G is not balanced and let H^* be a smallest (in length) unquad hole in G. By Remark 2.1 H^* is a smallest unquad hole in any signed graph obtained from G by a sequence of scalings. In this section we study properties of strongly adjacent nodes to H^*.

Definition 3.3 *A node u strongly adjacent to a hole H in G is* odd-strongly adjacent *if u has an odd number of neighbors in H. If u has an even number of neighbors in H, then u is* even-strongly adjacent. *The sets $A_r(H)$ and $A_c(H)$ contain the odd-strongly adjacent nodes in H which belong to V^r and V^c respectively.*

We will now prove the following fundamental properties of the sets $A_r(H^*)$ and $A_c(H^*)$ associated with a smallest unquad hole H^*.

Property 3.4 *Every even-strongly adjacent node to H^* is a twin of a node in H^*.*

Property 3.5 *There exists a node $x_r \in V^r \cap V(H^*)$ which is adjacent to all the nodes in $A_c(H^*)$.*

Property 3.6 *There exists a node $x_c \in V^c \cap V(H^*)$ which is adjacent to all the nodes in $A_r(H^*)$.*

Conforti and Rao [3] prove the above properties for a bipartite graph which is signed so that all of its edges have weight +1.

Proof of Property 3.4: Suppose u has an even number of neighbors, $u_1, u_2, \ldots, u_{2k}$, $k \geq 2$ in H^*. Let S_i, $i = 1, 2, \ldots, 2k$ be the sectors of (H^*, u) having nodes u_i, u_{i+1} as endnodes (where indices are taken modulo $2k$).

By scaling of the graph at every node u_i for which the edge uu_i has weight -1, we can obtain a graph in which all the spokes of (H^*, u) have weight +1. Now since H^* is unquad, there is a sector, say S_i, of weight 0 mod 4. Then the cycle u, u_i, S_i, u_{i+1}, u is an unquad hole of smaller length than H^*. Hence if u is an even-strongly adjacent node in H^* it must have exactly two neighbors, say u_1 and u_2. W.l.o.g the edges uu_1 and uu_2 have weight +1. Clearly the two u_1u_2-subpaths of H^* say P_1 and P_2, are such that one of them is of weight 0 mod 4 and the other is of weight 2 mod 4. Suppose P_2 is of weight 2 mod 4. Then P_2 must have length two for otherwise u, u_1, P_1, u_2, u would be an unquad hole of smaller length than H^* . Hence u_1 and u_2 must have a common neighbor, say u^*, in H^*. □

To prove Property 3.5 and Property 3.6 we need the following lemma.

Lemma 3.7 *If $u, v \in A_c(H^*)$, then they have at least one common neighbor in H^*. Moreover in any sector of (H^*, v), node u has either an even number of neighbors, or exactly one neighbor adjacent to v.*

Proof: First we show that u cannot have an odd number, greater than one, of neighbors in any one sector of (H^*, v). Suppose not. Let u have an odd number of neighbors, greater than one in sector S_k of (H^*, v). Let $H = v, S_k, v$. Now (H, u) is an odd wheel, therefore this wheel contains an unquad hole which must be of smaller length than H^*. Hence u must have either an even number or exactly one neighbor in any sector of (H^*, v).

Next we show that if node u has exactly one neighbor in some sector then this node is also adjacent to v. This in turn implies that at least one node in H^* is a neighbor of both u and v since node u has an odd number of neighbors in H^*.

Suppose in sector S_k node u has a unique neighbor u_k which is not a neighbor of v. Let v_{k-1} and v_k be the end nodes of S_k, P_1 and P_2 be the $v_{k-1}u_k$ and v_ku_k-subpaths of S_k repectively. Since u is strongly adjacent to

H^*, it has a neighbor in another sector, say S_l having one endnode v_l distinct from v_{k-1} and v_k. Let u_l be the neighbor of u closest to v_l in sector S_l. Now there is a $3PC(u_k, v)$ using paths P_1, P_2 and nodes u_l and v_l. This 3-path configuration must contain an unquad hole which must be of smaller length than H^*, which contradicts our choice of H^*. $\square$

Lemma 3.8 *Every three nodes in $A_c(H^*)$ have a common neighbor in H^*.*

Proof: Let $U = \{u_1, u_2, u_3\} \subseteq A_c(H^*)$. Note that by Lemma 3.7 every pair of nodes in $A_c(H^*)$ have a common neighbor in H^*. Assume that there is no node of H^* that is adjacent to all three elements of U. Define the following sets :

$$\begin{aligned} A_{13} &= \{v \in V(H^*) | u_1 \textit{ and } u_3 \textit{ are adjacent to } v\} \\ A_{23} &= \{v \in V(H^*) | u_2 \textit{ and } u_3 \textit{ are adjacent to } v\} \\ A_{12} &= \{v \in V(H^*) | u_1 \textit{ and } u_2 \textit{ are adjacent to } v\} \end{aligned}$$

By our assumption $A_{12} \cap A_{23} = \phi$. Consider the wheel (H^*, u_1) and the strongly adjacent node u_3. Define $A^o_{13} = \{v \in A_{13}|$ in the two adjacent sectors of (H^*, u_1) with the common node v, there are in total an odd number of neighbors of $u_3\}$. (Note that this definition is not symmetric, i.e. A^o_{13} is not necessarily equal to A^o_{31}). Similarily define A^o_{23}. Now we prove the following two claims.

Claim 1: *Both A^o_{13} and A^o_{23} contain an odd number of elements.*

Proof of Claim 1: We prove that $|A^o_{13}|$ is odd. Consider the wheel (H^*, u_1) and let $S_1, \ldots, S_n$ be the sectors of this wheel, with S_i having endnodes s_i and s_{i+1} (where indices are taken modulo n). For every $i = 1, \ldots, n$ let x_i denote the number of neighbors of u_3 in sector S_i. By Lemma 3.7 every sector of (H^*, u_1) either has an even number of neighbors of u_3 or exactly one neighbor, in which case the neighbor is in A_{13}. This and the definition of A^o_{13} leads to the following properties:

(a) If $s_i \in A^o_{13}$ then either $x_{i-1} = x_i = 1$, or both x_{i-1} and x_i are even.

(b) If $s_i \in A_{13} \setminus A^o_{13}$ then either $x_{i-1} = 1$ and x_i is even, or x_{i-1} is even and $x_i = 1$.

(c) If s_i and s_{i+1} are not in A_{13} then x_i is even.

In the summation $\sum_{i=1}^{n} x_i$, every neighbor of u_3 which is in A_{13} is counted twice, so the total number of neighbors of u_3 on H^* is

$$|N(u_3) \cap V(H^*)| = \sum_{i=1}^{n} x_i - |A_{13}| \tag{1}$$

Further we will show that

$$\sum_{i=1}^{n} x_i \equiv |A_{13} \setminus A_{13}^o| \mod 2 \tag{2}$$

Now by (1) and (2) we have

$$\begin{aligned} |N(u_3) \cap V(H^*)| &\equiv (|A_{13} \setminus A_{13}^o| - |A_{13}|) \mod 2 \\ &\equiv -|A_{13}^o| \mod 2 \end{aligned}$$

Since u_3 is an odd-strongly adjacent node to H^*, we have that $|A_{13}^o|$ is odd.

Now we prove (2). Clearly the parity of $\sum_{i=1}^{n} x_i$ is the parity of the number of sectors with an odd number of neighbors of u_3. In this paragraph we will refer to these sectors as *odd sectors.* By Properties (a), (b) and (c), if S_i is an odd sector, then it has exactly one neighbor of u_3 (i.e. $x_i = 1$), and either s_i or s_{i+1} is an element of A_{13}. Each element in A_{13} belongs to $0, 1$ or 2 odd sectors. Clearly the parity of the number of odd sectors is equal to the parity of the number of elements in A_{13} which belong to exactly one odd sector. By Properties (a) and (c), $A_{13} \setminus A_{13}^o$ is the set of elements of A_{13} that belong to exactly one odd sector. Thus the parity of $\sum_{i=1}^{n} x_i$ is the same as the parity of $|A_{13} \setminus A_{13}^o|$.

This completes the proof of the claim.

Claim 2: *Let $v_1, v_2 \in V(H^*) \setminus A_{12}$ be neighbors of u_1 and u_2 respectively. If P is a v_1v_2-subpath of H^*, such that u_1 and u_2 have no neighbors in $V(P) \setminus \{v_1, v_2\}$, then u_3 has an even number of neighbors on P.*

Proof of Claim 2: Suppose that u_3 has an odd number of neighbors on P.

Case 1: u_3 has exactly one neighbor v_3 on P.

W.l.o.g $v_3 \neq v_1$. By Lemma 3.7, any two nodes of $A_c(H^*)$ have a common neighbor on H^*. Let $v_{12} \in V(H^*)$ be a common neighbor of u_1 and u_2, and let $v_{13} \in V(H^*)$ be a common neighbor of u_1 and u_3. By our assumption $A_{12} \cap A_{13} = \phi$, so $v_{12} \neq v_{13}$. Now there is a $3PC(v_3, u_1)$ where nodes v_1, v_{12}, v_{13} belong to distinct paths of the 3-path configuration, which must contain an unquad hole of length smaller than H^*. This contradicts our choice of H^*.

Case 2: u_3 has an odd number of neighbors, greater than one, on P.

Let v_{12} be defined as above. Now there is an odd wheel (C, u_3), where $C = u_1, v_1, P, v_2, u_2, v_{12}, u_1$. Since v_1 is an odd-strongly adjacent node either the v_1v_{12}-subpath of H^* that does not contain v_2 or the v_2v_{12}-subpath of H^* that does not contain v_1, is of length greater than two. Therefore the wheel contains an unquad hole of length smaller than H^*, which contradicts our choice of H^*. This completes the proof of Claim 2.

Now let $s_1, \ldots, s_n$ be the neighbors of u_1 on H^*, and $t_1, \ldots, t_m$ be the neighbors of u_2 on H^*. Let $P_1, \ldots, P_l$ be the subpaths of H^*, whose endnodes are consecutive elements of $\{s_1, \ldots, s_n, t_1, \ldots, t_m\}$ and are such that for every $i \in \{1, \ldots, l\}$, P_i and P_{i+1} (where indices are taken modulo l) have exactly one node in common. For every $i = 1, \ldots, l$, let x_i denote the number of neighbors of u_3 in P_i. Let the endnodes of P_i be denoted by p_i and p_{i+1} (where the indices are taken modulo l). By Lemma 3.7 and Claim 2 every P_i that does not have an even number of neighbors of u_3, has exactly one. The P_i's with exactly one neighbor of u_3 are characterized as follows:

(i) If $x_i = 1$ and $p_i \in A^o_{13}$, then by Claim 2, p_{i+1} is a neighbor of u_1. Now by Property (a) in Claim 1 $x_{i-1} = 1$ and hence by Claim 2, p_{i-1} is a neighbor of u_1. Similarily if $x_i = 1$ and $p_i \in A^o_{23}$, then $x_{i-1} = 1$ and both p_{i-1} and p_{i+1} are neighbors of u_2.

(ii) If $x_i = 1$ and $p_i \in A_{13} \setminus A^o_{13}$, then by Claim 2, p_{i+1} is a neighbor of u_1. Also either by Property (b) in Claim 1 or by Claim 2, x_{i-1} is even. Similarily if $x_i = 1$ and $p_i \in A_{23} \setminus A^o_{23}$, then p_{i+1} is a neighbor of u_2 and x_{i-1} is even.

In the summation $\sum_{i=1}^{n} x_i$, every neighbor of u_3 which is in $A_{13} \cup A_{23}$ is counted twice, so the total number of neighbors of u_3 on H^* is

$$|N(u_3) \cap V(H^*)| = \sum_{i=1}^{n} x_i - |A_{13}| - |A_{23}| \qquad (3)$$

Further we will show that

$$\sum_{i=1}^{n} x_i \equiv (|A_{13} \setminus A_{13}^o| + |A_{23} \setminus A_{23}^o|) \mod 2 \tag{4}$$

Now by (3) and (4) we have

$$\begin{aligned} |N(u_3) \cap V(H^*)| &\equiv (|A_{13} \setminus A_{13}^o| - |A_{13}| + |A_{23} \setminus A_{23}^o| - |A_{23}|) \mod 2 \\ &\equiv -(|A_{13}^o| + |A_{23}^o|) \mod 2 \end{aligned}$$

By Claim 1 $(|A_{13}^o| + |A_{23}^o|)$ is even, which contradicts our choice of u_3. Thus A_{13} and A_{23} cannot be disjoint.

Now we prove (4). Clearly the parity of $\sum_{i=1}^{n} x_i$ is the same as the parity of the number of sectors with an odd number of neighbors of u_3. If P_i has an odd number of neighbors of u_3, then it has exactly one neighbor (i.e. $x_i = 1$) and either p_i or p_{i+1} is an element of $A_{13} \cup A_{23}$. W.l.o.g. let $p_i \in A_{13}$. Pair off P_{i-1} and P_i if the only neighbor of u_3 in these paths is the node common to P_{i-1} and P_i, namely p_i. By Property (i) and (ii) this is possible if and only if $p_i \in A_{13}^o \cup A_{23}^o$. Notice that in this case $x_{i-1} + x_i = 2$ and the sectors together provide an even count in the sum $\sum_{i=1}^{n} x_i$. Hence the parity of $|A_{13} \cup A_{23}|$ is the same as the parity of $|A_{13} \setminus A_{13}^o| + |A_{23} \setminus A_{23}^o|$, and so (4) holds.

This completes the proof that A_{13} and A_{23} are not disjoint. Hence we have the proof of the lemma. □

Proof of Property 3.5: If H^* is of length 6 or less then the property clearly holds. Suppose now that H^* has length greater than 6. Suppose $W \subseteq A_c(H^*)$ is such that for every proper subset W' of W there exists a node of H^* which is adjacent to all nodes in W', but there exists no node of H^* adjacent to all nodes in W. By Lemma 3.7 and Lemma 3.8, $|W| > 3$. Let $W = \{w_i | i = 1, 2, \ldots, p\}$ and let $W_l = \{w_i | i = 1, \ldots, p, i \neq l\}$. Now for $l = 1, 2, \ldots, p$, all the nodes in W_l have a common neighbor say t_l, in H^*. Hence for $i = 1, \ldots, p$, node t_i is adjacent to w_j, for $j = 1, \ldots, p, j \neq i$, but t_i is not adjacent to w_i. Now there exists an odd wheel, $w_1, t_2, w_3, t_1, w_2, t_3, w_1$ with center t_4, hence it must contain an unquad hole smaller than H^*. This contradicts the choice of H^*. □

By symmetry Property 3.6 holds as well.

Lemma 3.9 *Let v be a twin of a node v_0 in H^*, with neighbors v_1 and v_2 in H^*. If H^* is of length greater than four, then the weights of the paths v_1, v, v_2 and v_1, v_0, v_2 are congruent modulo 4.*

Proof: Suppose not. Then the hole v_1, v_0, v_2, v, v_1 is unquad, and of smaller length than H^*, which contradicts our choice of H^*. □

Definition 3.10 *A* tent $\tau(H, u, v)$ *is a subgraph of G induced by node set $V(H) \cup \{u, v\}$, where H is a hole of G and u, v are adjacent nodes which are even-strongly adjacent to H with the following property:*

The nodes of H can be partitioned into two subpaths P_u and P_v containing the nodes in $N(u) \cap H$ and $N(v) \cap H$ respectively.

A tent $\tau(H, u, v)$ is referred to as a *tent containing H*. We now study properties of a tent $\tau(H^*, u, v)$ containing a smallest unquad hole H^* and we assume throughout the paper that the first node, say u in the definition of a tent $\tau(H, u, v)$ belongs to V^r and that node v belongs to V^c. We use the notation of Figure 1c, where nodes $u_1, u_0, u_2, v_1, v_0, v_2$ are encountered in this order, when traversing H^* counterclockwise, starting from u_1.

Lemma 3.11 *Nodes v_0, u_1, u_2 satisfy at least one of the following properties:*

- *The set $A_r(H^*)$ is contained in $N(v_0) \cup N(u_1)$.*
- *The set $A_r(H^*)$ is contained in $N(v_0) \cup N(u_2)$.*

Nodes u_0, v_1, v_2 satisfy at least one of the following properties:

- *The set $A_c(H^*)$ is contained in $N(u_0) \cup N(v_1)$.*
- *The set $A_c(H^*)$ is contained in $N(u_0) \cup N(v_2)$.*

Proof: We prove the first part. Suppose $w \in A_r(H^*)$ is not adjacent to v_0. Consider the hole H_1^* obtained from H^* by replacing v_0 with node v of $\tau(H^*, u, v)$. By Lemma 3.9, H_1^* is unquad, and since it is of the same length as H^*, it also is a smallest unquad hole. Now w cannot be adjacent to v, for otherwise w is even-strongly adjacent to H_1^*, which violates Property 3.4. Node u is in $A_r(H_1^*)$ and has neighbors u_1, u_2 and v in H_1^*. Since w is not adjacent to v, by Property 3.6 it follows that w is adjacent to u_1 or u_2.

Furthermore, by Property 3.6 the nodes in $A_r(H^*)$ which are not adjacent to v_0 are either all adjacent to u_1 or they are all adjacent to u_2. Therefore $A_r(H^*) \subseteq N(v_0) \cup N(u_1)$ or $A_r(H^*) \subseteq N(v_0) \cup N(u_2)$. The second part of the lemma can be proved similarly. □

Lemma 3.12 *Let $\tau(H^*, u, v)$ and $\tau(H^*, w, y)$ be two tents, where w_1, w_2 are the neighbors of w and y_1, y_2 are the neighbors of y in H^*. Let w_0 and y_0 be the common neighbors of w_1, w_2 and y_1, y_2 respectively. Then at least one of the following properties holds:*

- *Nodes u_1 and u_2 coincide with w_1 and w_2.*
- *Nodes v_1 and v_2 coincide with y_1 and y_2.*
- *Node u_0 coincides with y_1 or y_2.*
- *Node v_0 coincides with w_1 or w_2.*

Proof: Suppose the contrary. Then node u does not coincide with w, node v does not coincide with y , nodes u_0 and y are not adjacent and nodes v_0 and w are not adjacent. Let P denote the u_2v_1-subpath of H^* not containing any other neighbor of u or v. Similarly, let Q denote the v_2u_1-subpath of H^* not containing any other neighbors of u and v. Now it follows that y_1 and y_2 are contained in P or Q, and w_1 and w_2 are contained in P or Q. Assume w.l.o.g. that y_1 and y_2 are contained in P. We now prove the following two claims.

Claim 1: *Node y is not adjacent to u and node w is not adjacent to v.*

Proof of Claim 1: Suppose that y and u are adjacent. Now there is an odd wheel u_2, P, v_1, v, u, u_2 with center y. This wheel contains an unquad hole, which is by construction, of smaller length than H^*, which contradicts our choice of H^*. Hence y is not adjacent to u. By symmetry, it follows that w is not adjacent to v. This completes the proof of Claim 1.

Claim 2: *Nodes w_1 and w_2 belong to Q.*

Proof of Claim 2: Suppose not. Then w_1 and w_2 belong to P. By assumption, y_1 and y_2 belong to P. Let P' be the path obtained from P by substituting y for y_0. Now by Claim 1, there is an odd wheel u_2, P', v_1, v, u, u_2 with center w. This wheel contains an unquad hole, which is by construction,

of smaller length than H^*. This contradics our choice of H^*. Hence w_1 and w_2 belong to Q. This completes the proof of Claim 2.

Now by Claim 1 and Claim 2, there is a $3PC(u,y)$ that uses at most as many edges as there are in H^*. This 3-path configuration contains an unquad hole, of smaller length than H^*, which contradicts our choice of H^*. □

Definition 3.13 *A hole H is said to be* clean in G *if the following three conditions hold:*

(i) *No node is odd-strongly adjacent to H.*

(ii) *Every even-strongly adjacent node is a twin of a node in H.*

(iii) *There is no tent containing H.*

3.2 Induced Subgraphs Containing Clean Unquad Holes

In this section, we show how to create at most m^4n^4 induced subgraphs of G such that, if G is not balanced, one of the subgraphs, say G_t, contains a smallest unquad hole which is clean in G_t.

Definition 3.14 *Given a graph F, and a node $v \in V(F)$, we denote by $N_F(v)$ the set $N(v) \cap V(F)$.*

We define F_{ijkl} to be the induced subgraph of F obtained by removing the nodes in $N_F(j) \setminus \{i,k\}$ and the nodes in $N_F(k) \setminus \{j,l\}$.

PROCEDURE 2

Input: A signed bipartite graph G.

Output: A family $\mathcal{L} = \{G_1, G_2, \dots, G_p\}$, where $p \leq m^4n^4$, of induced subgraphs of G such that if G is not balanced, one of the subgraphs in $\mathcal{L}$, say G_t, contains a smallest unquad hole that is clean in G_t.

Step 1 Let $\mathcal{L}^* = \{G_{ijkl} \mid$ nodes i,j,k,l induce the chordless path i,j,k,l in $G\}$.

Step 2 Let $\mathcal{L} = \{Q_{ijkl} \mid$ the graph Q is in $\mathcal{L}^*$, nodes in $\{i,j,k,l\}$ belong to Q and induce the chordless path i,j,k,l of $Q\}$.

We now prove the validity of Procedure 2.

Lemma 3.15 *If G is not balanced, one of the graphs in $\mathcal{L}$, say G_t, contains an unquad hole H^*, smallest in G, and H^* is clean in G_t.*

Proof: Assume G is not balanced. Then G contains a smallest unquad hole H^*. Recall that the sets $A_r(H^*)$ and $A_c(H^*)$ are defined with respect to G. Consider the following two cases:

Case 1: There is no tent in G containing H^*.

By Property 3.5, there exists a node $j \in V^r(G) \cap V(H^*)$ that is a common neighbor of all nodes in $A_c(H^*)$. Let i, k be the neighbors of j in H^* and let l be the other neighbor of k in H^*. Then the graph G_{ijkl} contains H^*, but does not contain any node in $A_c(H^*)$, and belongs to $\mathcal{L}^*$. By considering G_{ijkl} and applying Property 3.6, it follows that $\mathcal{L}$ contains a graph G_t and H^* is clean in G_t.

Case 2: The graph G contains a tent $\tau(H^*, u, v)$.

By Lemma 3.11, the set $A_r(H^*)$ is contained in $N(v_0) \cup N(u_1)$ or in $N(v_0) \cup N(u_2)$ and the set $A_c(H^*)$ is contained in $N(u_0) \cup N(v_1)$ or in $N(u_0) \cup N(v_2)$. Assume w.l.o.g. that $A_r(H^*)$ is contained in $N(v_0) \cup N(u_1)$.

Suppose $A_c(H^*)$ is contained in $N(u_0) \cup N(v_1)$ and let u^* and v^* be the neighbors of u_1 and v_1, which are distinct from u_0 and v_0 respectively. By Lemma 3.11 and Lemma 3.12, it follows that the graph $G_{u^* u_1 u_0 u_2}$, which belongs to $\mathcal{L}^*$, contains H^* and satisfies the following properties:

- No node in $A_c(H^*)$ that is adjacent to u_0 belongs to $G_{u^* u_1 u_0 u_2}$.
- No node in $A_r(H^*)$ that is adjacent to u_1 belongs to $G_{u^* u_1 u_0 u_2}$.
- The graph $G_{u^* u_1 u_0 u_2}$ does not contain a node w, in a tent $\tau(H^*, w, y)$, where w_1 and w_2 coincide with u_1 and u_2.
- The graph $G_{u^* u_1 u_0 u_2}$ does not contain a node y, in a tent $\tau(H^*, w, y)$, where y and u_0 are adjacent.

As a consequence of Lemmas 3.11 and 3.12, applied to $G_{u^* u_1 u_0 u_2}$, it follows that $\mathcal{L}$ contains an induced subgraph of G, say G_t, which contains H^* and H^* is clean in G_t. If $A^c(H^*)$ is contained in $N(u_0) \cup N(v_2)$, the proof is identical. □

3.3 Double Star Decompositions

Definition 3.16 *A wheel with three spokes and at least two sectors of length 2 is said to be a* short 3-wheel.

In this section, we describe a procedure to decompose a signed bipartite graph with no short 3-wheel into blocks which are induced subgraphs and do not contain a double star cutset. While decomposing the graph into blocks, the procedure also checks the existence of a 3-path configuration that contains nodes in at least two connected components. But first we give a polynomial time procedure to check for the existence of a short 3-wheel.

PROCEDURE 1

Input: A signed bipartite graph G.
Output: A short 3-wheel of G or the fact that G does not contain such a node induced subgraph.

Step 1 Enumerate all distinct subsets of six nodes with three nodes in V^r and three nodes in V^c and declare them as unscanned. Go to Step 2.
Step 2 If all subsets are scanned, G does not contain a short 3-wheel, stop. Otherwise choose an unscanned subset U. If U induces a 6-cycle $C = a_1, a_2, a_3, a_4, a_5, a_6, a_1$, having unique chord a_2a_5, go to Step 3. Otherwise declare U as scanned and repeat Step 2.
Step 3 Remove the nodes in $N(a_2) \cup N(a_4) \cup N(a_5) \cup N(a_6) \setminus \{a_1, a_3\}$. If a_1 and a_3 are in the same connected component, then a short 3-wheel with spokes a_2a_1, a_2a_3, a_2a_5 is identified, stop. If not, remove the nodes in $N(a_1) \cup N(a_2) \cup N(a_3) \cup N(a_5) \setminus \{a_4, a_6\}$. If a_4 and a_6 are in the same connected component, then a short 3-wheel with spokes a_5a_2, a_5a_4, a_5a_6 is identified, stop. Otherwise declare U as scanned return to Step 2.

Remark 3.17 *The complexity of this procedure is of order $O(m^4n^4)$.*

Now we describe a procedure to perform double star decompositions.

PROCEDURE 3

Input: A signed bipartite graph F not containing a short 3-wheel or an unquad hole of length 4.

Output: Either a 3-path configuration is detected (hence F is not balanced) or a list of undominated signed induced subgraphs $F_1, \ldots, F_j, \ldots, F_q$ of F, where $q \leq |V^c(F)|^2|V^r(F)|^2 \leq m^2n^2$ is constructed with the following properties:

- The graphs $F_1, \ldots, F_j, \ldots, F_q$ do not contain a double star cutset.
- If the input graph F contains a clean unquad hole, then one of the graphs in the list, say F_i, contains an unquad hole of F which is clean in F_i.

Step 1 Delete dominated nodes in F until no such node exists. Let $\mathcal{M} = \{F\}, \mathcal{T} = \emptyset$.

Step 2 If $\mathcal{M}$ is empty, stop. Otherwise remove a graph R from $\mathcal{M}$. If R has no double star cutset, add R to $\mathcal{T}$ and repeat Step 2. Otherwise, let $S = N_R(u) \cup N_R(v)$ be a double star cutset of R. Let $R_1, \ldots, R_l$ be the connected components of $R \setminus S$, let $R_1^*, \ldots, R_l^*$ be the corresponding blocks, i.e. R_i^* is induced by $V(R_i) \cup S$. Go to Step 3.

Step 3 Consider every pair of nonadjacent nodes u_p and v_q such that node u_p is adjacent to u and node v_q is adjacent to v. If both u_p and v_q have neighbors in two distinct connected components of $R \setminus S$, there is a $3PC(u_p, v_q)$ and F is not balanced. Otherwise go to Step 4.

Step 4 From each block R_i^*, remove dominated nodes in $(N(u) \cup N(v)) \setminus \{u, v\}$, until no such node exists. Now remove further any dominated node until the block becomes undominated.

Add to $\mathcal{M}$ all the undominated blocks that contain at least one chordless path of length 3. Go to Step 2.

Remark 3.18 *If a node $w \in (N(u) \cup N(v)) \setminus \{u, v\}$ belongs to the undominated block R_i^* at the end of Step 4, then w is adjacent to at least one node in the connected component R_i.*

Before proving the validity of Procedure 3, we need the following definition:

Definition 3.19 *Let G be a signed bipartite graph containing a hole H. Then $C(H) = \{H_i \mid H_i$ is obtained from H by a sequence of holes $H = H_0, H_1, \ldots, H_i$, where H_j and H_{j-1}, for $j = 1, 2, \ldots, i$, differ in one node $\}$.*

Lemma 3.20 *Let G be a signed bipartite graph which contains no unquad holes of length 4. Let H be an unquad hole in G. If H' and H differ in at most one node, then H' is unquad.*

Proof: Let H' be obtained from H by replacing node u by node v. Let x and y be the common neighbors of u and v in H. Since G contains no unquad of length four, the paths x, u, y and x, v, y have the same weight modulo 4. Thus, H' is unquad. □

Lemma 3.21 *Let G be a signed bipartite graph containing a smallest unquad hole H^*, but not containing a short 3-wheel and not containing an unquad hole of length 4. If H^* is clean in G, then every hole H_i^* in $C(H^*)$ is clean in G.*

Proof: We prove the lemma by induction: it suffices to show that, if H_1^* is a hole that differs from H^* in only one node, then H_1^* is clean in G.

By Lemma 3.20, H_1^* is an unquad hole of smallest length. By Property 3.4, condition (ii) of Definition 3.13 is satisfied. Hence, if the lemma is false, condition (i) or (iii) of Definition 3.13 is not satisfied. Therefore we consider the following two cases.

Case 1: Condition (i) of Definition 3.13 is not satisfied.

Now a node w must be odd-strongly adjacent to H_1^*. Since no node is odd-strongly adjacent to H^*, it follows that w has three neighbors, say w_1, w_2, w_3 in H_1^*. Two of these neighbors, say w_1 and w_2 must be in H^* and, by Property 3.4, they have a common neighbor, say w_0 in H^*. Since w_3 is in H_1^* but not in H^*, it follows that H_1^* is obtained from H^* by replacing some node $u \neq w_1$, w_2 in H^* with w_3. Let u_1 and u_2 be the neighbors of u in H^*. Note that w_3 is adjacent to u_1 and u_2 and u does not coincide with w_1 or w_2. Hence u_1 and u_2 do not coincide with w_0. Now $\tau(H^*, w_3, w)$ is a tent, contradicting the assumption that H^* is clean in G.

Case 2: Condition (iii) of Definition 3.13 is not satisfied.

There must be a tent $\tau(H_1^*, u, v)$. We first show the following claim:

Claim: *At least one of the nodes u_1, u_2, v_1, v_2 does not belong to the hole H^*.*

Proof of Claim: Assume not. Since u and v are not in H_1^*, it follows that at most one of them is in H^*. If u is in H^*, then u_0 is not in H^* and v is odd-strongly adjacent to H^*. So u is not in H^* and, by symmetry, node v is not in H^*.

Assume that neither u nor v belong to H^* and let $w \neq u_1, u_2, v_1, v_2$ be a node in H^* but not in H_1^*. Nodes w and u are not adjacent, otherwise node u is odd-stongly adjacent to H^*, contradicting the assumption that H^* is clean. By symmetry, it follows that nodes w and v are not adjacent. Now $\tau(H^*, u, v)$ is a tent, contradicting the assumption that H^* is clean and the proof of the claim is complete.

By the above claim, one of the nodes u_1, u_2, v_1, v_2 is not in H^*. Assume w.l.o.g. that u_2 is not in H^*. Clearly, node u is not in H^*. Node v is not in H^*, otherwise node v_0 is not in H^*, node u_2 coincides with v_0 and $\tau(H_1^*, u, v)$ is not a tent.

Thus the hole H_1^* is obtained from H^* by replacing a node w with u_2, where w is adjacent to u_0. Let u_3 in H^* be the other neighbor of u_2. It follows that u_3 is adjacent to w. Let Q denote the v_1u_3-subpath of H^* not containing v_2. Consider the hole $C = u, v, v_1, Q, u_3, w, u_0, u_1, u$. Now the wheel (C, u_2) is a short 3-wheel, contradicting the fact that G does not contain a short 3-wheel. □

Remark 3.22 *Assume that the signed bipartite graph F contains a smallest unquad hole H^* that is clean in F. If F does not contain a short 3-wheel and it does not contain an unquad hole of length 4, then an undominated graph obtained from F by deleting all the dominated nodes contains a clean unquad hole in the family $C(H^*)$.*

Lemma 3.23 *Let F be a signed bipartite graph satisfying the following properties:*

- *The graph F does not contain a short 3-wheel.*
- *The graph F does not contain an unquad hole of length 4.*
- *The graph F contains a smallest unquad hole H^* that is clean in F.*

Then the output of Procedure 3 is one of the following:

- *A 3-path configuration is detected in Step 3.*
- *One of the undominated blocks, say F_i, obtained as ouput of Procedure 3, contains an unquad hole in $C(H^*)$.*

Proof: Let $S = N(u) \cup N(v)$ be a double star cutset of F. Let $F_1, \ldots, F_t$ be the connected components of $F \setminus S$ and $F_1^*, \ldots, F_t^*$ be the corresponding blocks. We now show that an unquad hole $H' \in \mathcal{C}(H^*)$ is contained in some block F_i^* obtained at the end of Step 3. There are three cases to consider.

Case 1: Both nodes u and v belong to H^*.

Let u_1 and v_1 in H^* be the other neighbors of u and v respectively. Now the nodes in $V(H^*) \setminus \{u, v, u_1, v_1\}$ are in some connected component F_i and F_i^* contains H^*.

Case 2: Either node u or node v is in H^*.

Assume w.l.o.g. that u is in H^* and v is not in H^*. Let u_1 and u_2 be the neighbors of u in H^*. Note that v can have at most one neighbor distinct from u in H^*. Suppose v does not have any neighbor other than u in H^*. Then the nodes in the set $V(H^*) \setminus \{u, u_1, u_2\}$ are in some connected component F_i and F_i^* contains H^*. Suppose v has one other neighbor, say v_1, in H^*. Now v_1 and u must have a common neighbor, say u_1, in H^*. Now the nodes in the set $V(H^*) \setminus \{v_1, u, u_1, u_2\}$ are in some connected component F_i and it follows that F_i^* contains H^*.

Case 3: Neither u nor v belongs to H^*.

Assume w.l.o.g. that $|N(u) \cap V(H^*)| \leq |N(v) \cap V(H^*)|$ There are three subcases to consider:

Case 3.1: The set $N(u) \cap V(H^*)$ is empty.

If $|N(v) \cap V(H^*)| = 0$ or 1, the unquad hole H^* is preserved in some block F_i^*. Suppose now that $N(v) \cap V(H^*) = \{v_1, v_2\}$. Let v_0 be the common neigbor of v_1 and v_2 in H^*. Now the nodes in $V(H^*) \setminus \{v_0, v_1, v_2\}$ will be in some connected component F_i. If v_0 is in F_i, then the block F_i^* contains H^*. If v_0 is not in F_i, let H'' be obtained from H^* by replacing v_0 with v. Now H'' belongs to $\mathcal{C}(H^*)$ and the block F_i^* contains H''.

Case 3.2: $N(u) \cap V(H^*) = \{u_1\}$.

Now $|N(v) \cap V(H^*)| = 1$ or 2. Suppose $N(v) \cap V(H^*) = \{v_1\}$. If u_1 and v_1 are adjacent in H^*, then H^* is preserved in some block F_i^*. Suppse u_1 and v_1 are not adjacent. Let P and Q be the two u_1v_1-subpaths of H^*. The nodes in $V(P) \setminus \{u_1, v_1\}$ will be in some connected component F_i and the nodes in $V(Q) \setminus \{u_1, v_1\}$ will be in some connected component F_j. If the two connected components coincide, H^* is preserved in F_i^*. If the two connected components do not coincide, there is a $3PC(u_1, v_1)$ and Step 3 in Procedure 3 detects this 3-path configuration.

Suppose $N(v) \cap V(H^*) = \{v_1, v_2\}$. Let v_0 be the common neighbor of v_1 and v_2 in H^*. Scale at v_1 and v_2 to get the edges vv_1 and vv_2 to have weight $+1$. Now since F does not contain an unquad hole of length 4, the weight of the path v_1, v_0, v_2 is congruent to 2 mod 4. Now scale at u and u_1 to get the edges uv and uu_1 to have weight $+1$. Let P be the u_1v_1-subpath of H^* that does not contain v_2, and let Q be the u_1v_2-subpath of H^* that does not contain v_1. $w(P)$ and $w(Q)$ are either congruent to 1 or 3 mod 4. Since $w(v_1, v_0, v_2) \equiv 2 \bmod 4$, $w(P) \not\equiv w(Q) \bmod 4$. Now if u_1 is not adjacent to v_1 or v_2, then either v, u, u_1, P, v_1, v or v, u, u_1, Q, v_2, v is unquad and of smaller length than H^*. Suppose u_1 and v_1 are adjacent. Now the nodes in $V(H^*) \setminus \{u_1, v_1, v_0, v_2\}$ will be in some connected component F_i. If v_0 is in the same connected component F_i then H^* is preserved in F_i^*. Suppose v_0 is not in the same connected component F_i. Let H'' be obtained from H^* by replacing v_0 with v. Now H'' belongs to $\mathcal{C}(H^*)$ and the block F_i^* contains H''.

Case 3.3: $N(u) \cap V(H^*) = \{u_1, u_2\}$.

Now $N(v) \cap V(H^*) = \{v_1, v_2\}$. Let u_0 be the common neighbor of u_1 and u_2 in H^* and let v_0 be the common neighbor of v_1 and v_2 in H^*. If u_0 is not adjacent to v and v_0 is not adjacent to u there is a tent $\tau(H^*, u, v)$. So assume w.l.o.g. that u_0 coincides with v_1. Then v_2 is adjacent to u_2 and H^* is preserved in some block F_i^*.

Thus in all cases some block F_i^* contains the unquad hole H^* or an unquad hole H'' in $\mathcal{C}(H^*)$. Now by Lemma 3.21 the unquad hole H'' is clean in F and hence H'' clean in F_i^*. By Remark 3.22 the undominated graph F_i^* defined in Step 4 of Procedure 3 must contain an unquad hole in $\mathcal{C}(H^*)$. Repeating the same argument for every undominated block F_i^*, which contains an unquad hole in the family $\mathcal{C}(H^*)$ and is added to the list $\mathcal{M}$, the lemma follows. □

Lemma 3.24 *The number of induced subgraphs in the list $\mathcal{T}$ produced by Procedure 3 is bounded by $|V^c(F)|^2|V^r(F)|^2$.*

Proof: Let $S = N(u) \cup N(v)$ be a double star cutset of F. Let $F_1, \dots, F_t$ be the connected components of $F \setminus S$ and let $F_1^*, \dots, F_t^*$ be the corresponding undominated blocks. We prove the following two claims.

Claim 1: *No two distinct undominated blocks contain the same chordless path of length 3.*

Proof of Claim 1: Suppose by contradiction that a chordless path $P = a, b, c, d$ belongs to two distinct undominated blocks F_i^* and F_j^*. Then $\{a, b, c, d\} \subseteq N_F(u) \cup N_F(v)$. There are three cases to consider.

Case 1: Both nodes u and v belong to $\{a, b, c, d\}$.

Node d cannot coincide with u for otherwise a and d are adjacent and P is not a chordless path. Similarly d does not coincide with v and a does not coincide with u or v. Hence we can assume that $u = b$ and $v = c$. From Step 4 of Procedure 3 it follows that node a has at least one neighbor in each of the connected components F_i and F_j for otherwise it would have been deleted from one or both the undominated blocks F_i^* and F_j^*. Similarly node d has at least one neighbor in each of the connected components F_i and F_j. Now Step 3 of Procedure 3 detects a 3-path configuration.

Case 2: Either u or v belongs to $\{a, b, c, d\}$.

The same argument used in Case 1 shows that node u coincides with b or c. Assume w.l.o.g. that u and b coincide. Now a and c are neighbors of u, d is adjacent to v and both a and d must have at least one neighbor in F_i and F_j. Again Step 3 of Procedure 3 detects a 3-path configuration.

Case 3: Both u and v do not belong to $\{a, b, c, d\}$.

As in the previous cases both a and d must have at least one neighbor in F_i, at least one neighbor in F_j and Step 3 of Procedure 3 detects a 3-path configuration. This completes the proof of Claim 1.

Claim 2: *The graph F contains at least one chordless path of length 3 which is not contained in any of the undominated blocks F_i^*.*

Proof of Claim 2: Each of the connected components $F_1, \ldots, F_t$ must contain at least two nodes, since F is an undominated graph. At least one node in F_i must be adjacent to a node in $N_F(u) \cup N_F(v)$. Assume w.l.o.g. that node p_i in F_i is adjacent to a neighbor of v, say d_i. Suppose now no node in F_i is adjacent to a node in $N(u)$. Then by Step 4 of Procedure 3, the undominated block F_i^* does not contain any neighbor of u other than v. This in turn implies that in the same step node u would have been deleted from F_i^*. Now $P = p_i, d_i, v, u$ is a chordless path of length 3 in F but P is not in any of the undominated blocks $F_1^*, \ldots, F_t^*$. So a node in F_i must be adjacent to a node, say s_i, which is a neighbor of u. Repeating the same argument for $j = 1, \ldots, t$, it follows that each connected component F_j contains a node, say w_j, which is adjacent to a node, say $s_j \in N_F(u)$. Suppose now s_j has a neighbor, say g in a connected component F_k, distinct from F_j. Let q be

a neighbor of g in F_k. Then $P = q, g, s_j, w_j$ is a chordless path of length 3 which is contained in F but not in any of the undominated blocks $F_1^*, \ldots, F_t^*$.

Suppose now that s_j does not have any neighbor in a connected component, say F_l. Then in Step 4 of Procedure 3, node s_j is deleted from the undominated block F_l^*. Now the path w_l, s_l, u, s_j is a chordless path of length 3 which is contained in F but not in any of the undominated blocks $F_1^*, \ldots, F_t^*$. This completes the proof of Claim 2.

Every undominated block that is added to the list $\mathcal{M}$ in Step 4 of Procedure 3 contains a chordless path of length 3. Hence every undominated block that is added to the list $\mathcal{T}$ in Step 2 contains a chordless path of length 3. By Claim 1, the same chordless path of length 3 is not in any other undominated block that is added to the list $\mathcal{T}$. By Claim 2, it follows that the number of double star cutsets used to decompose the graph F with Procedure 3 is at most $|V^c(F)|^2|V^r(F)|^2$. Hence the lemma follows. □

4 6-Join Decompositions

In this section we describe a procedure to decompose a signed bipartite graph into blocks that do not contain a 6-join. We also show that if the graph does not contain an extended star cutset then neither do the undominated blocks.

PROCEDURE 4

Input: A signed bipartite graph G, not containing an unquad hole of length 4 or 6, or a short 3-wheel.

Output: A list of signed bipartite graphs $\mathcal{M} = \{D_1, D_2, \ldots, D_r\}$, satisfying the following properties:

- No graph in the list $\mathcal{M}$ contains a 6-join.
- The graph G is balanced if and only if all the graphs in the list $\mathcal{M}$ are balanced.

Step 1 Let $\mathcal{L} = \{G\}$, and $\mathcal{M} = \emptyset$.

Step 2 If $\mathcal{L} = \emptyset$, stop. Otherwise remove a graph R from $\mathcal{L}$. Enumerate all distinct subsets of six nodes with three nodes in $V^r(R)$ and three nodes in $V^c(R)$ and declare them as unscanned. Go to Step 3.

Step 3 If all six node subsets are scanned, add R to $\mathcal{M}$ and return to Step 2. Otherwise choose an unscanned subset U and declare it scanned. If the nodes in U do not induce a 6-hole $a_1, a_2, \ldots, a_6, a_1$ in R, then repeat Step 3. Otherwise, let $A_j = \{a_j\}$ for every $j = 1, \ldots, 6$, $T = \{a_1, a_3, a_5\}$ and $B = \{a_2, a_4, a_6\}$. Let $S = V(R) \setminus (T \cup B)$ and go to Step 4.

Step 4 Apply to the nodes in S, the following rules in order, repeatedly, until no further application is possible.

Rule 1: If u is adjacent to at least one node in each of A_i, A_{i+2}, A_{i+4}, where i is odd, then if u is adjacent to a node in B then go to Step 3, else put u in T and remove it from S.

Rule 2: If u is adjacent to at least one node in each of A_i, A_{i+2}, A_{i+4}, where i is even, then if u is adjacent to a node in T then go to Step 3, else put u in B and remove it from S.

Rule 3: If u is adjacent to a node in A_i, where i is odd, but not to any node node in $A_{i+2} \cup A_{i+4}$, then if u is adjacent to a node in B then go to Step 3, else put u in T and remove it from S.

Rule 4: If u is adjacent to a node in A_i, where i is even, but not to any node node in $A_{i+2} \cup A_{i+4}$, then if u is adjacent to a node in T then go to Step 3, else put u in B and remove it from S.

Rule 5: If u is adjacent to a node in A_i and a node in A_{i+2}, where i is odd, and u is adjacent to a node in T, then if u is also adjacent to a node in B then go to Step 3, else put u in T and remove it from S.

Rule 6: If u is adjacent to a node in A_i and a node in A_{i+2}, where i is even, and u is adjacent to a node in B, then if u is also adjacent to a node in T then go to Step 3, else put u in B and remove it from S.

Rule 7: If u is adjacent to a node in B, a node in A_i and a node in A_{i+2}, where i is odd, then if there exists a node in $A_i \cup A_{i+2}$ to which u is not adjacent, then go to Step 3, else put u in A_{i+1} and in B and remove it from S.

Rule 8: If u is adjacent to a node in T, a node in A_i and a node in A_{i+2}, where i is even, then if there exists a node in $A_i \cup A_{i+2}$ to which u is not adjacent, then go to Step 3, else put u in A_{i+1} and in T and remove it from S.

Rule 9: If u is adjacent to a node in A_i and a node in A_{i+2}, where i is odd, but u is not adjacent to some node in $A_i \cup A_{i+2}$, then if u is also adjacent to a node in B then go to Step 3, else put u in T and remove it from S.

Rule 10: If u is adjacent to a node in A_i and a node in A_{i+2}, where i is even, but u is not adjacent to some node in $A_i \cup A_{i+2}$, then if u is also adjacent to a node in T then go to Step 3, else put u in B and remove it from S.

Rule 11: If u is not adjacent to any node in $\bigcup_{i=1}^{6} A_i$, but it is adjacent to a node in T, then if u is also adjacent to a node in B then go to Step 3, else put u in T and remove it from S.

Rule 12: If u is not adjacent to any node in $\bigcup_{i=1}^{6} A_i$, but it is adjacent to a node in B, then if u is also adjacent to a node in T then go to Step 3, else put u in B and remove it from S.

Step 5 Remove all nodes in S that are adjacent to every node in $A_2 \cup A_6$ and put them in A_1 and in T. Remove all nodes in S that are adjacent to every node in $A_2 \cup A_4$ and put them in A_3 and in T. Remove all nodes in S that are adjacent to every node in $A_4 \cup A_6$ and put them in A_5 and in T. Now $G(\cup_{i=1}^{6} A_i)$ defines a 6-join that separates T from B.

Step 6 Construct the blocks R_1 and R_2. Delete all dominated nodes and add the blocks to $\mathcal{L}$. Return to Step 2.

Remark 4.1 *The rules in Step 4 of Procedure 4 are forcing in the sense that if any of them holds, either node u must be removed from S and added to one of the sets $T, B, A_1, A_2, A_3, A_4, A_5, A_6$ if there is a 6-join, or it is detected that no 6-join is possible. In Step 5 of Procedure 4 the nodes that remain in S are of the following two types:*

- *a node u is not adjacent to any node in $T \cup B$*
- *a node u is adjacent to every node in $A_i \cup A_{i+2}$, for some i, but it is not adjacent to any node in $(T \cup B) \setminus (A_i \cup A_{i+2})$.*

Now by Step 5 it follows that $G(\cup_{i=1}^{6} A_i)$ defines a 6-join. Moreover the graphs in list $\mathcal{M}$ do not contain a 6-join.

Lemma 4.2 *Let G be a signed bipartite graph not containing an extended star cutset, a short 3-wheel, and not containing an unquad hole of length 4 or 6. Let $\mathcal{M} = \{D_1, D_2, \dots, D_r\}$ be the list of graphs produced from G by Procedure 4. Then r is $O(n + m)$ and the graphs in $\mathcal{M}$ do not contain an extended star cutset or a 6-join. Moreover G is balanced if and only if all the graphs in the list $\mathcal{M}$ are balanced.*

Proof: Let G be a signed bipartite graph, not containing an extended star cutset, or a short 3-wheel, or an unquad hole of length 4 or 6, that is decomposed by Procedure 4. Suppose $A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$ is a 6-join of G that separates G_1 (which contains A_i, i odd) from G_2 and let G_1^* and G_2^* be the corresponding blocks obtained in Step 6 (after deleting all dominated nodes). We now show that G_1^* and G_2^* do not contain an extended star cutset. Suppose G_1^* contains an extended star cutset $S = (x; X; Y; N)$.

Case 1: a_2, a_4 or a_6 is an isolated node in the graph $G \setminus S$.

W.l.o.g. let a_2 be isolated. Then $A_1 \cup A_3 \subset S$, which implies there is a node in G_1^* which dominates a_2. But then a_2 would have been deleted from G_1^*.

Case 2: $y, z \in V(G_1^*) \setminus \{a_2, a_4, a_6\}$ are such that y and z belong to separate components in $G_1^* \setminus S$.

Now we will construct from S an extended star $S^* = (x^*, X^*, Y^*, N^*)$ in the original graph. If any of a_2, a_4, a_6 are in $S \setminus X$ then replace them by the corresponding sets A_2, A_4, A_6 in the original graph. Let $X^* = X \setminus \{a_2, a_4, a_6\}$. If a_2, a_4 or a_6 is x, then add the corresponding set A_2, A_4 or A_6 to X^* and label one of the nodes from the set x^*. If a_2 is in $X \setminus \{x\}$, then if Y contains at least two nodes from A_1, let Y^* contain exactly these nodes, add the nodes in $(Y \setminus Y^*) \cup N$ to N^*, and add A_2 and A_6 to X^*. If Y contains at least two nodes from A_3 but not from A_1, let Y^* contain exactly these nodes, add the nodes in $(Y \setminus Y^*) \cup N$ to N^*, and add sets A_2 and A_4 to X^*. Otherwise add A_2 to X^*. Perform similar modifications to S to obtain S^* if $a_4 \in X \setminus \{x\}$ or $a_6 \in X \setminus \{x\}$.

By the above construction S^* is an extended star.

Claim: *S^* is an extended star cutset in the original graph.*

Proof of Claim: Assume that S^* is not an extended star cutset in the original graph. Then there exists a path P in $G \setminus S^*$ which connects y and z. This path must use two nodes $a'_i \in A_i$ and $a'_j \in A_j$ where $i \neq j$ and i and j are even. W.l.o.g. let us assume it uses $a'_2 \in A_2$ and $a'_4 \in A_4$. But then a_2 and a_4 are in different components in $G_1^* \setminus S$. W.l.o.g. let there exist a path from a_2 to y and from a_4 to z in $G_1^* \setminus S$. Since a_2 and a_4 are not connected $A_3 \subset S$. Also one of either $A_1 \subset S$ or $A_5 \subset S$ or $a_6 \in S$. But a_6 can only be in S if it is in $X \setminus \{x\}$ since it is not adjacent to any node in A_3. If Y contains at least one node from each of A_1 and A_5, then x is the center of a short 3-wheel. Thus, Y contains two nodes from either A_1 or A_5, then by the construction of S^* one of either A_4 or A_2 is also in X^*. But then a'_2 and a'_4

cannot be connected in $G \setminus S^*$ which contradicting the existence of P. This completes the proof of the Claim.

But the above claim contradicts our assumption that G did not contain an extended star cutset.

Hence G_1^* does not contain an extended star cutset. By symmetry, G_2^* does not contain an extended star cutset. Now repeating the same argument for every graph that is added to the list $\mathcal{L}$, it follows that every graph in the list $\mathcal{M}$ produced by Procedure 4 does not contain an extended star cutset. By Remark 4.1, the graphs in the list $\mathcal{M}$ do not contain a 6-join. Now Remark 3.22 and a repeated application of Theorem 2.7 shows that if G is balanced, all the graphs in the list $\mathcal{M}$ are balanced and if G is not balanced at least one graph in the list $\mathcal{M}$ is not balanced.

In order to complete the proof of the lemma we now show that the number of graphs in the list $\mathcal{M}$ is $O(n+m)$. This is seen by observing that in each 6-join decomposition the sum of the nodes in the two blocks is exactly 6 more than the number of nodes in the original graph. This completes the proof of the lemma. □

5 2-Join Decompositions

In this section we describe a procedure to decompose a signed bipartite graph G into blocks that do not contain a 2-join. We also show that if G does not contain an extended star cutset or a 6-join then neither do the final blocks.

PROCEDURE 5

Input: A signed bipartite graph G not containing an unquad hole of length 4.

Output: A list of signed bipartite graphs $\mathcal{N} = \{B_1, B_2, \ldots, B_r\}$, where r is $O(n+m)$, satisfying the following properties:

- No graph in the list $\mathcal{N}$ contains a 2-join.
- The graph G is balanced if and only if all the graphs in the list $\mathcal{N}$ are balanced.

Step 1 Let $\mathcal{L} = \{G\}$, and $\mathcal{N} = \emptyset$.

Step 2 If $\mathcal{L} = \emptyset$, stop. Otherwise remove a graph R from $\mathcal{L}$. Enumerate all distinct subsets of four nodes $c_1, c_2 \in V^c$, $r_1, r_2 \in V^r$ such that c_1r_1 and c_2r_2 are edges but c_1r_2 and c_2r_1 are not. Declare this set of four nodes as unscanned. Go to Step 3.

Step 3 If all subsets of four nodes in $V(R)$ are scanned, add R to $\mathcal{N}$ and return to Step 2. Otherwise choose an unscanned subset $\{c_1r_1, c_2r_2\}$ and go to Step 4.

Step 4 Define $A = \{c_1\}$, $B = \{r_1\}$, $D = \{c_2\}$, $F = \{r_2\}$. Apply Procedure 6 to check whether there exists a 2-join $E(K_{A'B'}) \cup E(K_{D'F'})$, where $A \subseteq A'$, $B \subseteq B'$, $D \subseteq D'$, $F \subseteq F'$. If no such 2-join exists, go to Step 5. If a 2-join has been identified, construct the blocks R_1^* and R_2^*, add them to the list $\mathcal{L}$ and return to Step 2.

Step 5 Define $A = \{c_1\}$, $B = \{r_1\}$, $D = \{r_2\}$, $F = \{c_2\}$. Apply Procedure 6 to check whether there exists a 2-join $E(K_{A'B'}) \cup E(K_{D'F'})$, where $A \subseteq A'$, $B \subseteq B'$, $D \subseteq D'$, $F \subseteq F'$. If no such 2-join exists, declare U as scanned and return to Step 3. If a 2-join has been identified, construct the blocks R_1^* and R_2^*, add them to the list $\mathcal{L}$ and return to Step 2.

PROCEDURE 6

Input: A bipartite graph R and node disjoint bicliques K_{AB} and K_{DF} such that no node in A is adjacent to a node in D and no node in B is adjacent to a node in F.

Output: Either a 2-join $E^* = E(K_{A'B'}) \cup E(K_{D'F'})$, where $A \subseteq A'$, $B \subseteq B'$, $D \subseteq D'$, $F \subseteq F'$ is identified, or no such 2-join exists.

Step 1 Let $S = \emptyset$ and $T = V(R) \setminus (A \cup B \cup D \cup F)$. Go to Step 2.

Step 2 Apply the Rules 1 to 11 to nodes in T repeatedly until no further application is possible.

Rule 1 If u is adjacent to a node in A and a node in F, there is no 2-join $E(K_{A'B'}) \cup E(K_{D'F'})$.

Rule 2 If u is adjacent to a node in B and a node in D, there is no 2-join $E(K_{A'B'}) \cup E(K_{D'F'})$.

Rule 3 If u is adjacent to a node in S, a node in B and a node in F, there is no 2-join $E(K_{A'B'}) \cup E(K_{D'F'})$.

Rule 4 If u is adjacent to a node in S and there exist two nodes $f_1, f_2 \in F$ such that u and f_1 are adjacent but u and f_2 are nonadjacent, there is no 2-join $E(K_{A'B'}) \cup E(K_{D'F'})$.

Rule 5 If u is adjacent to a node in S and there exist two nodes $b_1, b_2 \in B$ such that u and b_1 are adjacent but u and b_2 are nonadjacent, there is no 2-join $E(K_{A'B'}) \cup E(K_{D'F'})$.

Rule 6 If u is adjacent to a node in A and a node in D, remove u from T and add u to S.

Rule 7 If u is not adjacent to any node in $A \cup B$ and there exist two nodes $d_1, d_2 \in D$ such that u and d_1 are adjacent but u and d_2 are nonadjacent, remove u from T and add it to S.

Rule 8 If u is not adjacent to any node in $D \cup F$ and there exist two nodes $a_1, a_2 \in A$ such that u and a_1 are adjacent but u and a_2 are nonadjacent, remove u from T and add it to S.

Rule 9 If u is adjacent to all nodes in F and to at least one node in S, but u is not adjacent to any node in B, remove u from T and add it to D.

Rule 10 If u is adjacent to all nodes in B and to at least one node in S, but u is not adjacent to any node in F, remove u from T and add it to A.

Rule 11 If u is adjacent to at least one node in S, but u is not adjacent to any node in $B \cup F$, remove u from T and add it to S.

Step 3 Remove from T every node u that is adjacent to all nodes in A and add u to B. Remove from T every node v that is adjacent to all nodes in D and add v to F. Let $A' = A$, $B' = B$, $D' = D$ and $F' = F$. Now $E(K_{A'B'}) \cup E(K_{D'F'})$ defines a 2-join, separating $A' \cup D' \cup S$ from $B' \cup F' \cup T$.

Lemma 5.1 *After Step 2 of Procedure 6 no node in T is adjacent to a node in S, and if a node $u \in T$ is adjacent to a node in $A \cup D$ then u is one of the following two types:*

(i) u is adjacent to every node in A, but no node in $D \cup F$, or

(ii) u is adjacent to every node in D and no node in in $A \cup B$.

Proof: Rules 3,4,5 and 11 characterize all nodes that are in T and adjacent to a node in S. So after Step 2 of Procedure 6 has been completed no node in T is adjacent to a node in S. By Rules 1 and 6, if a node $u \in T$ is adjacent to a node in A, then it is not adjacent to any node in $D \cup F$. Now by Rule 8 u is adjacent to every node in A. Similarily, by Rules 2 and 6, if a node $u \in T$ is adjacent to a node in D, then it is not adjacent to any node in $A \cup B$. Then by Rule 7 u must be adjacent to all nodes in D. □

Remark 5.2 *The rules in Step 2 of Procedure 6 are forcing in the sense that if any of them holds, node u must be removed from T and added to one of the sets A, D or S if there is a 2-join $E(K_{A'B'}) \cup E(K_{D'F'})$, where $A \subseteq A'$, $B \subseteq B'$, $D \subseteq D'$, $F \subseteq F'$. Rules 1 to 5 detect a contradiction that arises as a consequence of removing u from T and adding to one of the sets A, D or S. Now by Lemma 5.1 and Step 3 it follows that the bicliques identified by Procedure 6 define a 2-join. Moreover the graphs in the list $\mathcal{N}$ do not contain a 2-join.*

Lemma 5.3 *Let G be a signed bipartite graph not containing an extended star cutset, a 6-join and not containing an unquad hole of length 4. Let $\mathcal{N} = \{B_1, B_2, \ldots, B_r\}$ be the list of graphs produced from G by Procedure 5. Then r is of $O(n+m)$ and the graphs in $\mathcal{N}$ do not contain an extended star cutset, a 6-join or a 2-join. Moreover if G is balanced all the graphs in the list $\mathcal{N}$ are balanced and if G is not balanced at least one graph in the list $\mathcal{N}$ is not balanced.*

Proof: Let G be a signed bipartite graph, not containing an extended star cutset or a 6-join, that is decomposed by Procedure 5. Suppose $E^* = E(K_{AB}) \cup E(K_{DF})$ is a 2-join of G that separates G_1 from G_2 and let G_1^* and G_2^* be the corresponding blocks.

Notice that the blocks contain no holes of length less than 7, which use the paths P_{ad} and P_{bf}. Hence if the original graph did not contain a 6-join, neither can the two blocks.

We now show that G_1^* and G_2^* do not contain an extended star cutset. Suppose G_1^* contains an extended star cutset $S = (x; X; Y; N)$. Let the nodes in A and D belong to G_1 and let nodes b and f in G_1^* represent the nodes in B and F respectively. The nodes b and f are connected by a path P_{bf} which is of length 4 or 5. There are four cases to consider.

Case 1: Node x coincides with b or f.

Assume w.l.o.g. that x coincides with b. Since P_{bf} is of length at least 4 and E^* defines a 2-join, it follows that node f and the nodes in D are not in S. Hence S separates the nodes in D from a node in $G_1 \setminus A$. If $X = \{x\}$, then S is a star cutset of G_1^* separating the nodes in D from a node in $G_1 \setminus A$. Now every node in B defines a star cutset of G separating the nodes in D from a node in $G_1 \setminus A$. Hence X must contain at least two nodes. Then at least two nodes in A are contained in Y. Let x^* be a node in B . Let $N^* = N_G(x^*) \setminus Y$

and $X^* = (X \setminus \{x\}) \cup B$. Now $S^* = (x^*, X^*, Y, N^*)$ defines an extended star cutset of G separating the nodes in D from a node in $G_1 \setminus A$.

Case 2: Node x is an intermediate node of P_{bf}.

At least one of the nodes b or f is not in S since P_{bf} is of length at least 4. Assume w.l.o.g. that node f is not in S. Now S is a star cutset of G_1^* separating the nodes in D from a node in $G_1 \setminus A$. Then node b must be a star cutset of G_1^* separating the nodes in D from a node in $G_1 \setminus A$ and we are in Case 1.

Case 3: Node x is in A or in D.

Assume w.l.o.g. that x is in A. Now node $f \notin X$ since E^* defines a 2-join. Then S is an extended star cutset of G_1^* separating f from a node in $G_1 \setminus S$. If node b is not in S, it follows that S is an extended star cutset of G separating the nodes in F from a node in $G_1 \setminus S$. Suppose now node b is in S. Then if b is in N, let $N^* = (N \setminus \{b\}) \cup B$. Now $S^* = (x, X, Y, N^*)$ is an extended star cutset of G separating the nodes in F from a node in G_1. Otherwise b is in Y, which means that $X \subseteq A$. Let $Y^* = (Y \setminus \{b\}) \cup B$, and now $S^* = (x, X, Y^*, N)$ is an extended star cutset of G seperating the nodes in F from a node in G_1.

Case 4: Node x is in G_1 but not in $A \cup D$.

Now node b or f is not in S. Assume w.l.o.g. that f is not in S. Then S is an extended star cutset of G_1^* separating node f from a node in $G_1 \setminus S$. If node b is not in S it follows that S is an extended star cutset of G separating the nodes in F from a node in $G_1 \setminus S$. Suppose now node b is in S. Then b must be in X and $Y \subseteq A$ and it must contain at least two nodes. Let $X^* = (X \setminus \{b\}) \cup B$. Now $S^* = (x, X^*, Y, N)$ is an extended star cutset of G separating the nodes in F from a node in G_1.

Hence G_1^* does not contain an extended star cutset or a 6-join. By symmetry, G_2^* does not contain an extended star cutset or a 6-join. Now repeating the same argument for every graph that is added to the list $\mathcal{L}$, it follows that every graph in the list $\mathcal{N}$ produced by Procedure 5 does not contain an extended star cutset or 6-join. By Remark 5.2, the graphs in the list $\mathcal{N}$ do not contain a 2-join. Since none of the graphs created in the intermediate steps of Procedure 5 contain a biclique cutset, a repeated application of Theorem 2.4 and Remark 2.3 shows that if G is balanced, all the graphs in the list $\mathcal{N}$ are balanced and if G is not balanced at least one graph in the list $\mathcal{N}$ is not balanced.

In order to complete the proof of the lemma we now show that the number of graphs in the list $\mathcal{N}$ is of $O(n+m)$. This is easily seen by observing that in each 2-join decomposition the sum of the number of nodes in the two blocks is at most 12 more than the number of nodes in the original graph. If we stop doing 2-join decompositions when the size of the blocks is smaller than 24 then the number of blocks created is only linear in the number of nodes in the original graph. This completes the proof of the lemma. $\Box$

6 Recognition Algorithm and its Validity

We now give the recognition algorithm, prove its validity and polynomial time bound.

ALGORITHM

Input: A signed bipartite graph G.
Output: The signed graph G is identified as balanced or not balanced.

Step 1 Check whether G contains an unquad hole of length 4 or 6. Apply Procedure 1 to check whether G contains a short 3-wheel. If so, G is not balanced, otherwise go to Step 2.

Step 2 Apply Procedure 2 to create at most m^4n^4 induced subgraphs of G, say $G_1, \ldots, G_i, \ldots, G_p$ such that, if G is not balanced, at least one of the induced subgraphs created, say G_i, contains an unquad hole of smallest length which is clean in G_i.

Step 3 Apply Procedure 3 to each of the induced subgraphs $G_1, \ldots, G_i, \ldots, G_p$ to decompose them into undominated induced subgraphs $F_1, \ldots, F_j, \ldots, F_q$ that do not contain a double star cutset. While decomposing a graph with a double star cutset $N(u) \cup N(v)$, Procedure 3 also checks the existence of a 3-path configuration containing nodes u and v and nodes in two distinct connected components resulting from the decomposition. If such a 3-path configuration is found, then G is not balanced, otherwise go to Step 4.

Step 4 Apply Procedure 4 to each of the induced subgraphs $F_1, \ldots, F_j, \ldots, F_q$ to decompose them into undominated induced subgraphs $D_1, \ldots, D_k, \ldots, D_r$ that do not contain an extended star cutset or a 6-join. Go to Step 5.

Step 5 Apply Procedure 5 to each of the subgraphs $D_1, \ldots, D_k, \ldots, D_r$ to decompose them using 2-joins into blocks $B_1, \ldots, B_l, \ldots, B_s$ not containing an extended star cutset, 6-join or a 2-join.

Step 6 Test whether any of the blocks $B_1, \ldots, B_l, \ldots, B_s$ that are not R_{10} contains an unquad cycle. If so, then the signed graph G is not balanced, otherwise G is balanced.

Remark 6.1 *An algorithm to test whether a signed bipartite graph contains an unquad cycle can be found in [4] or [6]. Hence the details of Step 6 are omitted in this paper.*

Theorem 6.2 *The running time of the algorithm described in Section 3 is bounded from above by a polynomial function of the cardinalities m and n of the node sets V^r and V^c respectively. Moreover the algorithm correctly identifies a signed bipartite graph G as balanced or not.*

Proof: The running time of each of the procedures in the algorithm has been shown in its respective section to be bounded from above by a polynomial function of m and n. Testing wether a block is R_{10} can be done in constant time. The algorithms in [4] and [6], to check whether a signed bipartite graph contains an unquad cycle, are bounded from above by a polynomial function of m and n. Hence the running time of the algorithm described in Section 3 is bounded from above by a polynomial function of m and n.

Suppose G is balanced. Clearly G cannot contain a short 3-wheel or a 3-path configuration. All the induced subgraphs of G are balanced and the graphs produced by Procedures 2 and 3 are balanced. Consequently, by Lemma 4.2 and by Lemma 5.3, all the graphs in the final list $\mathcal{N}$ produced by Procedure 5 are balanced and do not contain an extended star cutset, a 6-join, or a 2-join. Now by Theorem 2.2 every graph in the list $\mathcal{N}$ does not contain an unquad cycle. Then Step 5 of the algorithm identifies G as balanced.

Suppose G is not balanced. If G contains a short 3-wheel, Step 1 of the algorithm identifies G as not balanced. Suppose G does not contain a short 3-wheel. Clearly the signed bipartite graph G contains an unquad hole of smallest length. Now by Lemma 3.15 one of the induced subgraphs of G, say G_i, in the list produced by Procedure 2 contains an unquad hole H^*, of smallest length, which is clean in G_i. Now G_i is one of the graphs considered for double star decompositions by Procedure 3. By Lemma 3.23, Procedure 3 either detects a 3-path configuration or one of the undominated blocks, say F, in the final list produced by Procedure 3 contains an unquad hole in the

family $\mathcal{C}(H^*)$. In the former case clearly G is not balanced. In the latter case Procedures 4 and 5 preserve a clean unquad hole in the graph. Now by Lemma 4.2 and Lemma 5.3 one of the blocks, say B_j, produced by Procedure 5 is not balanced. Clearly the block B_j contains an unquad hole and hence an unquad cycle. Hence Step 5 of the algorithm identifies G as not balanced. This completes the proof of the theorem. □

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